(#) Odrediti jednačinu tangentne ravni na površ $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, koja je normalna na pravu $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$.

Rj. Kako izgleda jednačina tangentne ravni i normale na neku površ $F(x,y,z)=0$?

$$F'_x(p_1,p_2,p_3)(x-p_1)+F'_y(p_1,p_2,p_3)(y-p_2)+F'_z(p_1,p_2,p_3)(z-p_3)=0$$

jednačina tangentne ravni na površ u tački $M(p_1,p_2,p_3)$

$$\frac{x-p_1}{F'_x(p_1,p_2,p_3)}=\frac{y-p_2}{F'_y(p_1,p_2,p_3)}=\frac{z-p_3}{F'_z(p_1,p_2,p_3)}$$ jednačina normale na površ $F(x,y,z)=0$ u tački $M(p_1,p_2,p_3)$

U našem slučaju $F(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$, a tačka u kojoj trebamo postaviti datu ravan nam nije poznata. Za trenutak, označimo tu tačku sa $M(x_0,y_0,z_0)$.

$$F'_x=\frac{2x}{a^2},\quad F'_y=\frac{2y}{b^2},\quad F'_z=\frac{2z}{c^2}$$

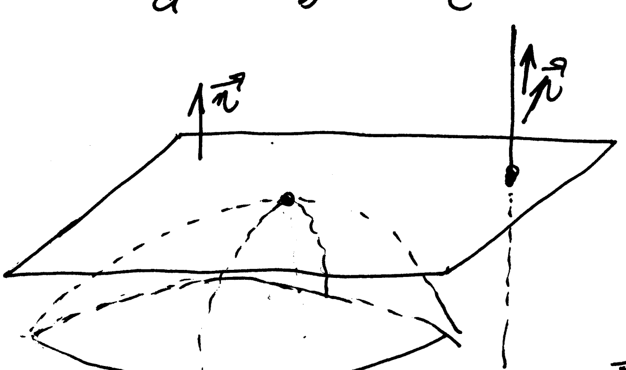
$$F'_x(M)(x-x_0)+F'_y(M)(y-y_0)+F'_z(M)(z-z_0)=0$$

$$\frac{2x_0}{a^2}(x-x_0)+\frac{2y_0}{b^2}(y-y_0)+\frac{2z_0}{c^2}(z-z_0)=0 \quad /:2$$

$$\frac{x_0}{a^2}x+\frac{y_0}{b^2}y+\frac{z_0}{c^2}z-\left(\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}+\frac{z_0^2}{c^2}\right)=0$$

Kako je tačka $M(x_0,y_0,z_0)$ tačka na elipsi imamo $\frac{x_0^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

tj. $$\frac{x_0}{a^2}x+\frac{y_0}{b^2}y+\frac{z_0}{c^2}z-1=0.$$



Vektor normale tražene ravni je $\vec{n}\left(\frac{x_0}{a^2},\frac{y_0}{b^2},\frac{z_0}{c^2}\right)$.

$\vec{n}\parallel\vec{p}$ gdje je $\vec{p}=(1,2,3) \Rightarrow \vec{n}=k\cdot\vec{p}$

$\Rightarrow \vec{n}=(k,2k,3k)$, $k\in R$ tj. imamo

$$\frac{x_0}{a^2}=k,\quad \frac{y_0}{b^2}=2k,\quad \frac{z_0}{c^2}=3k \Rightarrow x_0=a^2k,\ y_0=2kb^2,\ z_0=3kc^2$$

Postavljamo još pitanje kako izračunati k?
M je tačka sa naše površi (sa elipse) pa

$$\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}+\frac{z_0^2}{c^2}=1 \quad \text{tj.} \quad a^2k^2+4b^2k^2+9c^2k^2=1$$

$$k^2=\frac{1}{a^2+4b^2+9c^2}$$

$$k=\frac{\pm 1}{\sqrt{a^2+4b^2+9c^2}}$$

Na kraju imamo

$$\frac{x_0}{a^2}x+\frac{y_0}{b^2}y+\frac{z_0}{c^2}z-1=0$$

$$kx+2ky+3kz-1=0 \quad /:k$$

$$x+2y+3z-\frac{1}{k}=0$$

$$x+2y+3z\mp\sqrt{a^2+4b^2+9c^2}=0$$ jednačine tražene tangentne ravni

Izračunati integral po glatkom luku koji spaja tačke A i B

$$\int_{\widehat{AB}} \left(1-\frac{1}{y}+\frac{y}{z}\right)dx+\left(\frac{x}{z}+\frac{x}{y^2}\right)dy-\frac{xy}{z^2}\,dz$$

$A(1,1,1)$, $B(1,2,3)$, $\widehat{AB}\subseteq\{(x,y,z)\mid x>0,\ y>0,\ z>0\}$.

Rj. Označimo sa $P(x,y,z)=1-\frac{1}{y}+\frac{y}{z}$, $Q(x,y,z)=\frac{x}{z}+\frac{x^2}{y^2}$,

$R(x,y,z)=-\frac{xy}{z^2}$, i izračunajmo $\frac{\partial^2 P}{\partial y\partial z}$, $\frac{\partial^2 Q}{\partial x\partial z}$ i $\frac{\partial^2 R}{\partial x\partial y}$

$$\frac{\partial P}{\partial y}=-(-1)y^{-2}+\frac{1}{z} \qquad \frac{\partial Q}{\partial x}=\frac{1}{z}+\frac{1}{y^2} \qquad \frac{\partial R}{\partial x}=-\frac{y}{z^2}$$

$$\frac{\partial^2 P}{\partial y\partial z}=-\frac{1}{z^2} \qquad \frac{\partial^2 Q}{\partial x\partial z}=-\frac{1}{z^2} \qquad \frac{\partial^2 R}{\partial x\partial y}=-\frac{1}{z^2}$$

Kako je $\frac{\partial^2 P}{\partial y\partial z}=\frac{\partial^2 Q}{\partial x\partial z}=\frac{\partial^2 R}{\partial x\partial y}$ to integral ne zavisi od vrste krive linije koja spaja tačke A i B.

Odredimo f-ju $u=u(x,y,z)$ za koju vrijedi da je

$$du=\left(1-\frac{1}{y}+\frac{y}{z}\right)dx+\left(\frac{x}{z}+\frac{x}{y^2}\right)dy-\frac{xy}{z^2}\,dz$$

$$\frac{\partial u}{\partial x}=1-\frac{1}{y}+\frac{y}{z};$$

$$u=\int\left(1-\frac{1}{y}+\frac{y}{z}\right)dx+\varphi(y,z)$$

$$u=x-\frac{x}{y}+\frac{xy}{z}+\varphi(y,z)$$

$$\frac{\partial u}{\partial y}=\frac{x}{y^2}+\frac{x}{z}+\varphi'_y(y,z)$$

$$\frac{\partial u}{\partial y}=\frac{x}{y^2}+\frac{x}{z}$$

$$\varphi'_y(y,z)=0$$

$$\varphi(y,z)=C+\psi(z) \quad \ldots(1)$$

$$\frac{\partial u}{\partial z}=-\frac{xy}{z^2}+\varphi'_z$$

$$\frac{\partial u}{\partial z}=-\frac{xy}{z^2}$$

$$\varphi'_z=0 \quad \ldots(2)$$

(1) i (2) $\Rightarrow \psi(z)=0 \Rightarrow \varphi(y,z)=C$

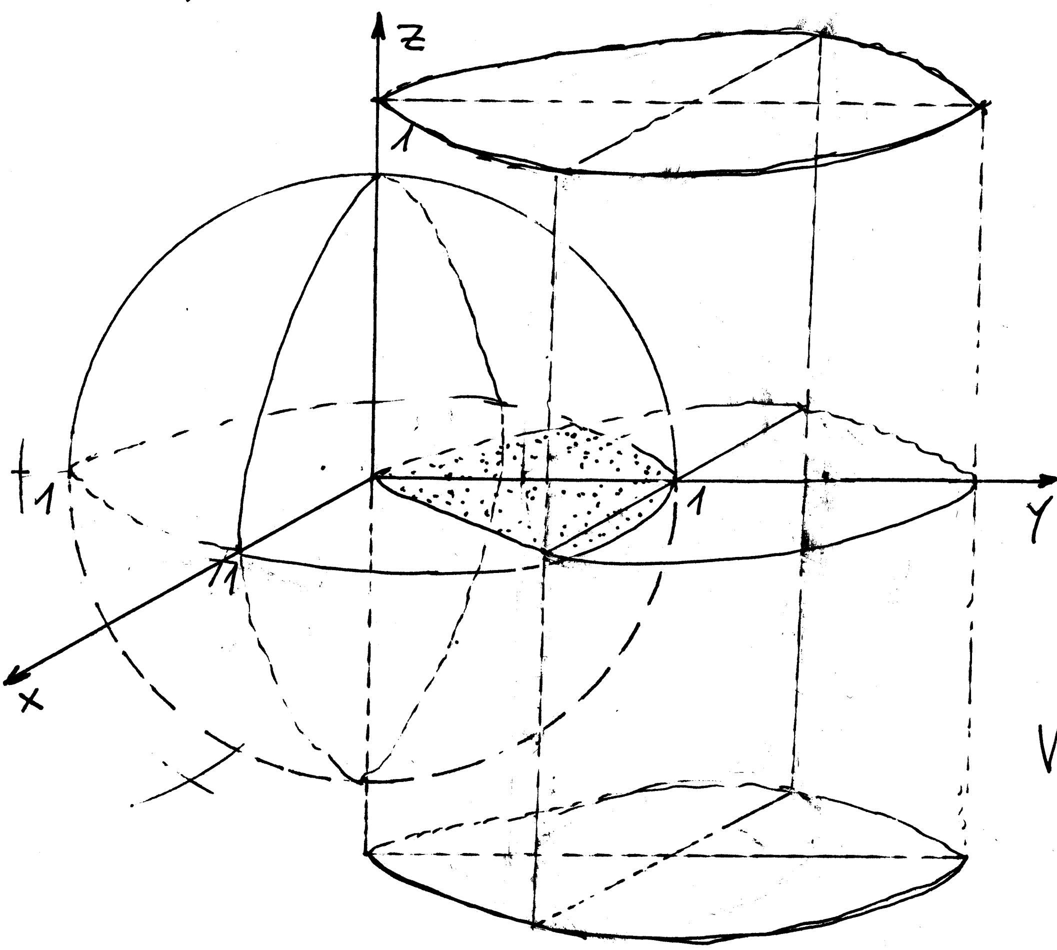
$$u=x-\frac{x}{y}+\frac{xy}{z}+C$$

$$\int_{\widehat{AB}}\left(1-\frac{1}{y}+\frac{y}{z}\right)dx+\left(\frac{x}{z}+\frac{x}{y^2}\right)dy-\frac{xy}{z^2}\,dz=\int_{\widehat{AB}}du=\left(x-\frac{x}{y}+\frac{xy}{z}\right)\Bigg|_{(1,1,1)}^{(1,2,3)}=1-\frac{1}{2}+\frac{2}{3}-1=\frac{1}{6}$$

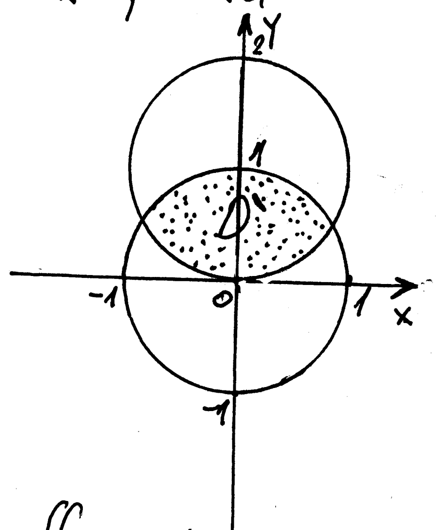
traženo rješenje

Izračunati zapreminu onog dijela lopte $x^2+y^2+z^2=1$ koji se nalazi unutar cilindra $x^2+(y-1)^2=1$.

Rj. Nacrtajmo skicu ove dvije figure u prostoru



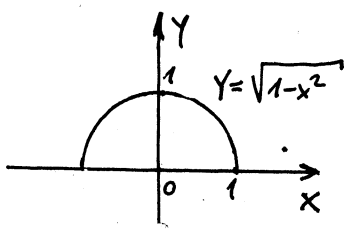
projekcije na xOy osu

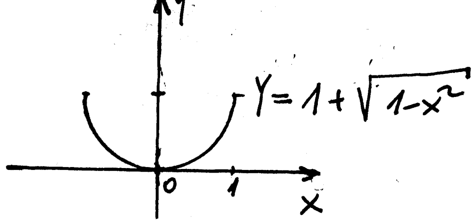


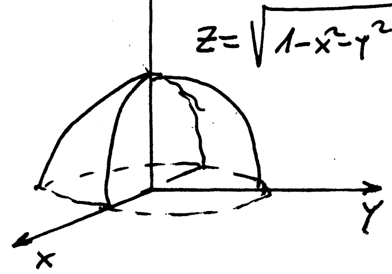
$$V=\iint_D z(x,y)\,dx\,dy$$

Zapremina tijela koje je odozgo ograničeno sa površi z a čija je projekcija na xOy ravan oblast D

Primjetimo da je presjek cilindra i lopte prvo simetričan u odnosu na xOy osu, a drugo da je simetričan u odnosu na yOz osu.







$$\frac{1}{4}V=\int_0^1 dx\int_{1+\sqrt{1-x^2}}^{\sqrt{1-x^2}}\sqrt{1-x^2-y^2}\,dy$$

$$\frac{1}{2}V=\iint_{D'} z(x,y)\,dx\,dy,\quad D':\begin{cases}-1\le x\le 1\\ 1+\sqrt{1-x^2}\le y\le\sqrt{1-x^2}\end{cases}$$

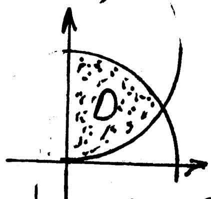
uvedimo polarne koordinate

$x=r\cos\varphi$

$y=r\sin\varphi$

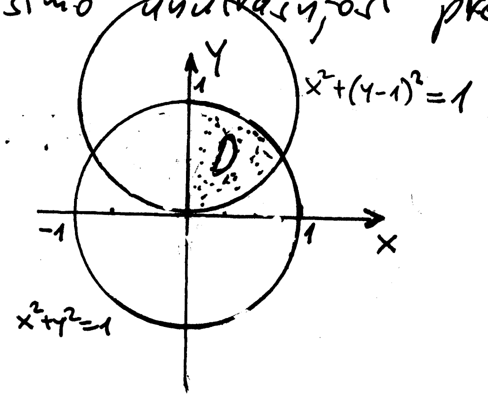
$dx\,dy=r\,dr\,d\varphi$

Kako opisati oblast



uz pomoć polarnih koordinata?

Opišimo unutrašnjost presjeka dva kruga pomoću polarnih koordinata.



$$x^2+y^2 \le 1$$
$$(r\cos\varphi)^2+(r\sin\varphi)^2 \le 1$$
$$r^2 \le 1$$
$$0 \le r \le 1$$

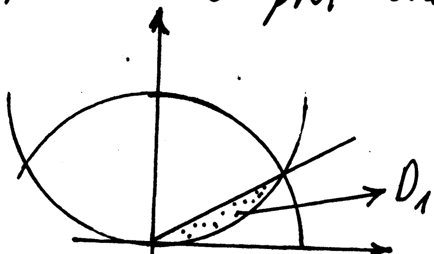
$$x^2+(y-1)^2 \le 1$$
$$x^2+y^2-2y+1 \le 1$$
$$x^2+y^2 \le 2y$$
$$r^2 \le 2r\sin\varphi \quad /:r$$
$$r \le 2\sin\varphi$$
$$0 \le r \le 2\sin\varphi$$

Kako je $0 \le \sin\varphi \le 1$ (ako posmatramo prvi kvadrant) to je moguć i slučaj da je $2\sin\varphi > 1$ pa imamo dva slučaja

1° $2\sin\varphi \le 1 \Rightarrow \sin\varphi \le \frac{1}{2}$ (pa ako posmatramo prvi kvadrant) ($\sin\varphi \ge 0$)

$\Rightarrow \varphi \in (0, \frac{\pi}{6})$

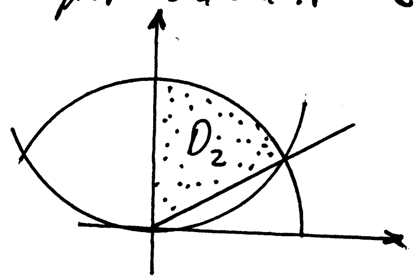
$$D_1: \begin{cases} 0 \le \varphi \le \frac{\pi}{6} \\ 0 \le r \le 2\sin\varphi \end{cases}$$



2° $2\sin\varphi \ge 1 \Rightarrow \sin\varphi \ge \frac{1}{2}$ (pa za prvi kvadrant $\sin\varphi \le 1$)

$\Rightarrow \varphi \in (\frac{\pi}{6}, \frac{\pi}{2})$

$$D_2: \begin{cases} \frac{\pi}{6} \le \varphi \le \frac{\pi}{2} \\ 0 \le r \le 1 \end{cases}$$



$D = D_1 \cup D_2$

$$\frac{1}{4}V = \iint_D \sqrt{1-r^2}\, r\,dr\,d\varphi = \iint_{D_1} r\sqrt{1-r^2}\,dr\,d\varphi + \iint_{D_2} r\sqrt{1-r^2}\,dr\,d\varphi$$

$$\iint_{D_1} r\sqrt{1-r^2}\,dr\,d\varphi = \int_0^{\frac{\pi}{6}} d\varphi \int_0^{2\sin\varphi} r\sqrt{1-r^2}\,dr = \left| \begin{array}{l} d(1-r^2) = \\ = -2r\,dr \end{array} \right| = \int_0^{\frac{\pi}{6}} d\varphi \int_0^{2\sin\varphi} \left(-\frac{1}{2}\right)\sqrt{1-r^2}\,d(1-r^2)$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{6}} \frac{(1-r^2)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^{2\sin\varphi} d\varphi = -\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}\int_0^{\frac{\pi}{6}} \left((1-\underbrace{4\sin^2\varphi}_{(2\sin\varphi)^2})^{\frac{3}{2}} - 1 \right) d\varphi$$

Ovo je eliptički integral i on se ne mora izračunati. Njegova približna vrijednost je $\pi/18$.

$$\iint_{D_2} r\sqrt{1-r^2}\,dr\,d\varphi = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} d\varphi \int_0^1 \left(-\frac{1}{2}\right)\sqrt{1-r^2}\,d(1-r^2) = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(-\frac{1}{2}\right)\frac{(1-r^2)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^1 d\varphi = -\frac{1}{3}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (-1)\,d\varphi$$

$$= \frac{1}{3}\left(\frac{\pi}{2}-\frac{\pi}{6}\right) = \frac{1}{3}\cdot\frac{3\pi-\pi}{6} = \frac{1}{3}\cdot\frac{\cancel{2}^{1}\pi}{\cancel{6}_{3}} = \frac{\pi}{9}$$

$$\frac{1}{4}V = \frac{\pi}{9}+\frac{\pi}{18} = \frac{3\pi}{18} = \frac{\pi}{6}$$

tražena zapremina

$$V = \frac{4\pi}{6} = \frac{2\pi}{3}$$

(#) Dato je vektorsko polje $\vec{A}=(e^x z-2xy,\ 1-x^2,\ e^x+z)$. Pokazati da je polje $\vec{A}$ potencijalno i odrediti mu potencijal. Izračunati integral $\int_L \vec{A}\cdot d\vec{r}$ gdje je L duž PQ, $P(0,1,-1)$, $Q(2,3,0)$ orjentisana od tačke P prema tački Q.

Rj. Ako je rotor vektorskog polja $\vec{A}$ jednak $\vec{0}$ ($\operatorname{rot}\vec{A}=\vec{0}$), tada za $\vec{A}$ kažemo da je <u>potencijalno polje</u>.

$$\operatorname{rot}\vec{A}=\nabla\times\vec{A}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ V_x & V_y & V_z\end{vmatrix}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ e^x z-2xy & 1-x^2 & e^x+z\end{vmatrix}$$

$$=(0-0)\vec{i}-(e^x-e^x)\vec{j}+(-2x+2x)\vec{k}=(0,0,0) \Rightarrow$$ $\vec{A}$ je potencijalno polje

F-ju $u=u(x,y,z)$ za koju vrijedi da je $\vec{A}=\operatorname{grad} u=\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}\right)$ zovemo <u>potencijal polja</u> $\vec{A}$. $\vec{A}=(e^x z-2xy,\ 1-x^2,\ e^x+z)$

$$u=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz$$

$$u=\int(e^x z-2xy)\,dx+\varphi(y,z)$$

$$u=e^x z-x^2 y+\varphi(y,z)$$

$$\frac{\partial u}{\partial y}=-x^2+\varphi'_y$$
$$\frac{\partial u}{\partial y}=1-x^2$$
$$\varphi'_y=1$$
$$\varphi(y,z)=y+\psi(z) \quad \ldots(1)$$

$$\frac{\partial u}{\partial z}=e^x+\varphi'_z$$
$$\frac{\partial u}{\partial z}=e^x+z$$
$$\varphi'_z=z \Rightarrow \varphi(y,z)=\frac{z^2}{2}+\psi(y) \quad \ldots(2)$$

$$(1)\ \text{i}\ (2) \Rightarrow \varphi(y,z)=y+\frac{z^2}{2}$$

Potencijal vektorskog polja $\vec{A}$ je $u=e^x z-x^2y+y+\frac{z^2}{2}+C$

$\int_L \vec{A}\cdot d\vec{r}$ zovemo <u>cirkulacija vektorskog polja</u> $\vec{A}$ duž krive L

$$C=\int_L \vec{A}\,d\vec{r}=\int_L V_x\,dx+V_y\,dy+V_z\,dz$$ gdje je $\vec{A}=(V_x,V_y,V_z)$, $d\vec{r}=(dx,dy,dz)$.

$$C=\int_L (e^x z-2xy)\,dx+(1-x^2)\,dy+(e^x+z)\,dz$$

ovo je krivolinijski integral druge vrste po krivoj datoj u prostoru

Prisjetimo se, ako je c kriva u ravni opisana parametarskim jednačinama $x=\eta(t)$, $y=\mu(t)$ gdje je $t_1\le t\le t_2$ tada krivolinijski integral se računa

$$\int_c P(x,y)\,dx+Q(x,y)\,dy=\int_{t_1}^{t_2}\Big(P(\eta(t),\mu(t))\,\eta'(t)+Q(\eta(t),\mu(t))\,\mu'(t)\Big)\,dt$$

Postavimo pravu kroz dvije date tačke $P(0,1,-1)$ i $Q(2,3,0)$.

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$ jednačina prave kroz dvije tačke

$P(\overset{x_1}{0},\overset{y_1}{1},\overset{z_1}{-1})$

$Q(\overset{x_2}{2},\overset{y_2}{3},\overset{z_2}{0})$

$$\frac{x-0}{2}=\frac{y-1}{2}=\frac{z+1}{1}\quad(=t)$$

$$L:\begin{cases}x=2t & dx=2\,dt\\ y=2t+1 & dy=2\,dt\\ z=t-1 & dz=dt\\ 0\le t\le 1\end{cases}$$

$$C=\int_0^1\Big(2\cdot\big(e^{2t}(t-1)-2\cdot 2t\cdot(2t+1)\big)+(1-4t^2)\cdot 2+\big(e^{2t}+(t-1)\big)\Big)\,dt$$

$$=\int_0^1\left(2e^{2t}t-2e^{2t}-16t^2-8t+2-8t^2+e^{2t}+t-1\right)dt$$

$$=\int_0^1 2e^{2t}t\,dt-\int_0^1 e^{2t}\,dt-24\int_0^1 t^2\,dt+\int_0^1(-7t+1)\,dt=\ldots=-\frac{19}{2}$$

$$2\int_0^1 e^{2t}t\,dt=\left|\begin{matrix}u=t & dv=e^{2t}dt\\ du=dt & v=\frac{1}{2}e^{2t}\end{matrix}\right|=2\frac{1}{2}te^{2t}\Big|_0^1-2\cdot\frac{1}{2}\int_0^1 e^{2t}\,dt=$$

$$=e^2-\frac{1}{2}e^{2t}\Big|_0^1=e^2-\frac{1}{2}e^2+\frac{1}{2}e^0=\frac{1}{2}e^2+\frac{1}{2}$$